J. Fluid Mech. (1971), *vol.* 49, *part* 4, *pp.* 745–751
Printed in Great Britain

On geostrophic motion of a non-homogeneous fluid

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(Received 22 June 1970 and in revised form 12 June 1971)

A general theoretical relationship between the (three-dimensional) velocity field and density field is established for geostrophic flow of a non-homogeneous fluid. The practical value of the relationship – which reduces to the celebrated two-dimensional theorem due to Proudman and Taylor when the fluid is homogeneous – is illustrated by means of three examples of flows in rapidly rotating fluids, namely (i) baroclinic waves in laboratory systems and in the atmosphere, (ii) the gyroscopic 'steering' processes in Jupiter's atmosphere that are implied by the 'Taylor column' theory of the Great Red Spot, and (iii) certain striking properties of ocean currents revealed by recent observations in the Atlantic and Mediterranean.

1. Introduction

The celebrated two-dimensional theorem due to Proudman (1916) and Taylor (1923) – that slow steady hydrodynamical motions of an inviscid homogeneous fluid that otherwise rotates like a rigid body are the same in all planes perpendicular to the rotation axis – has proved useful in a wide variety of studies. The purpose of the present paper is to establish the corresponding result for a non-homogeneous fluid. This is a problem of obvious interest in the theory of rotating fluids and one of practical importance in geophysical fluid dynamics.

2. Basic equations

When referred to a system of co-ordinates that rotates with uniform angular velocity $\mathbf{\Omega}$ relative to an inertial frame, the equations of motion and continuity governing the flow of a fluid of density ρ and viscosity μ may be conveniently written as follows:

$$2\mathbf{\Omega}\times\rho\mathbf{u}+\nabla p-\rho\mathbf{g} = -\mathbf{A}-\mathbf{R}-\mathbf{E} \tag{1}$$

and

$$\nabla.(\rho\mathbf{u}) = -C, \tag{2}$$

respectively, where

$$\mathbf{A}\equiv\rho\,\partial\mathbf{u}/\partial t,\quad \mathbf{R}\equiv\rho(\mathbf{u}.\nabla)\,\mathbf{u},\quad \mathbf{E}\equiv\nabla\times(\mu\nabla\times\mathbf{u})\quad\text{and}\quad C\equiv\partial\rho/\partial t. \tag{3}$$

Here $\mathbf{u}$ is the Eulerian flow velocity relative to the rotating system, p denotes pressure and t time, and $\mathbf{g}$ is the acceleration due to gravity and centrifugal effects. The terms **A**, **R** and **E** on the right-hand side of (1) can be neglected when Ω is so large that the dimensionless parameters

$$\mathfrak{A} \equiv \overline{\rho \frac{\partial \mathbf{u}}{\partial t}} \Big/ \overline{2\rho \boldsymbol{\Omega} \times \mathbf{u}}, \tag{4a}$$

$$\mathfrak{R} \equiv \overline{\rho(\mathbf{u}\,.\,\nabla)\,\mathbf{u}} / \overline{2\rho \boldsymbol{\Omega} \times \mathbf{u}} \quad \text{(Rossby number)} \tag{4b}$$

and

$$\mathfrak{E} \equiv \overline{\nabla \times (\mu \nabla \times \mathbf{u})} / \overline{2\rho \boldsymbol{\Omega} \times \mathbf{u}} \quad \text{(Ekman number)} \tag{4c}$$

(where the overbar means 'root-mean-square value of') tend to zero. The Coriolis force, $2\rho\boldsymbol{\Omega} \times \mathbf{u}$, then balances the non-hydrostatic component of the pressure gradient, $-\nabla p + \rho \mathbf{g}$, exactly and the flow is said to be *geostrophic*.

The equations of geostrophic flow are mathematically degenerate; they do not suffice, when combined with thermodynamic considerations (see equation (8)) and appropriate boundary conditions or initial conditions on $\mathbf{u}$, p and ρ, to determine the field of flow. Nevertheless, they express with good accuracy important properties that slow hydrodynamical motions in rapidly rotating systems (e.g. a wide range of laboratory experiments, the Earth's oceans and atmosphere, the atmospheres of Jupiter and certain other planets, certain stars) must possess *nearly* everywhere and can, when judiciously applied, lead to the discovery of the location and nature of highly ageostrophic phenomena (e.g. viscous or inertial boundary layers, detached shear layers ('fronts') or inertial oscillations) that are necessary concomitants of highly geostrophic flow.

Taking the curl of (1), making use of (2) and the fact that $\mathbf{g}$ is irrotational, we obtain the vorticity equation in the form

$$2\Omega \frac{\partial(\rho\mathbf{u})}{\partial s} - \mathbf{g} \times \nabla\rho = -2\boldsymbol{\Omega}C + \nabla \times (\mathbf{A} + \mathbf{R} + \mathbf{E}), \tag{5}$$

where $\partial/\partial s$ denotes differentiation with respect to distance s measured parallel to $\boldsymbol{\Omega}$. When $\mathfrak{A} \to 0$, $\mathfrak{R} \to 0$ and $\mathfrak{E} \to 0$ (the geostrophic limit, see equation (4)) and time variations in ρ are so slow that

$$\mathfrak{C} \equiv \overline{\frac{\partial \rho}{\partial t}} \Big/ \left\{ \overline{\frac{\partial(\rho u)}{\partial x}} + \overline{\frac{\partial(\rho v)}{\partial y}} + \overline{\frac{\partial(\rho w)}{\partial z}} \right\} \tag{6}$$

(see equation (2)) also tends to zero [where (u, v, w) are the (x, y, z) components of $\mathbf{u}$, the z axis being taken parallel to $\mathbf{g}$] equation (5) reduces to

$$2\Omega\, \partial(\rho\mathbf{u})/\partial s = \mathbf{g} \times \nabla\rho, \tag{7}$$

which expresses an exact balance between the gyroscopic torque, $2\Omega\,\partial(\rho\mathbf{u})/\partial s$, and the gravitational torque, $\mathbf{g} \times \nabla\rho$, acting on unit volume of an individual fluid element.

The density of an individual fluid element depends in general on the ambient pressure p, entropy X and chemical composition Y, so that the rate of change of density of the element, $D\rho/Dt$ (where $D/Dt \equiv \partial/\partial t + (\mathbf{u}\,.\,\nabla)$), satisfies

$$\frac{D\rho}{Dt} = \left(\frac{\partial\rho}{\partial p}\right)_{X,Y} \frac{Dp}{Dt} + \left(\frac{\partial\rho}{\partial X}\right)_{Y,p} \frac{DX}{Dt} + \left(\frac{\partial\rho}{\partial Y}\right)_{p,X} \frac{DY}{Dt}. \tag{8}$$

Now by (1)
$$\frac{Dp}{Dt} \equiv \frac{\partial p}{\partial t} + \mathbf{u}\,.\,\nabla p = \frac{\partial p}{\partial t} + \rho g w - \mathbf{u}\,.\,(\mathbf{A}+\mathbf{R}+\mathbf{E}). \tag{9}$$

Combining the last two equations shows that

$$u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\left(\frac{\partial \rho}{\partial z} - \rho\gamma^*\right) = -\frac{\partial \rho}{\partial t} + Q + \frac{1}{c^2}\left[\frac{\partial p}{\partial t} - \mathbf{u}\,.\,(\mathbf{A}+\mathbf{R}+\mathbf{E})\right], \tag{10}$$

where
$$\left.\begin{aligned} &c^2 \equiv (\partial\rho/\partial p)^{-1}_{X,Y}, \quad \gamma^* \equiv g/c^2 \\ \text{and}\quad &Q \equiv \left(\frac{\partial \rho}{\partial X}\right)_{p,Y}\frac{DX}{Dt} + \left(\frac{\partial \rho}{\partial Y}\right)_{p,X}\frac{DY}{Dt}, \end{aligned}\right\} \tag{11}$$

c being the speed of sound and $\rho\gamma^*$ the familiar adiabatic density gradient. The value of the quantity Q, which is of the order $\mathfrak{Q}$ times the largest terms on the left-hand side of equation (10) (the advective terms) if

$$\mathfrak{Q} \equiv \bar{Q}\bigg/\left\{\overline{u\frac{\partial \rho}{\partial x}} + \overline{v\frac{\partial \rho}{\partial y}} + \overline{w\left(\frac{\partial \rho}{\partial z} - \rho\gamma^*\right)}\right\}, \tag{12}$$

an inverse Péclet number, depends on irreversible thermodynamic processes (internal heating, thermal conduction, radiation) and processes that change the chemical composition of an individual fluid element. Q can be set equal to zero in the *isentropic* limit $\mathfrak{Q} \to 0$, when these irreversible processes are negligible. When, in addition, the fluid motions and the concomitant time variations in density and pressure are so slow that the dimensionless parameter

$$\mathfrak{D} \equiv \frac{\overline{\dfrac{\partial \rho}{\partial t}} + \overline{\dfrac{1}{c^2}\dfrac{\partial p}{\partial t}} + \overline{\dfrac{\mathbf{u}}{c^2}\,.\,(\mathbf{A}+\mathbf{R}+\mathbf{E})}}{\overline{u\dfrac{\partial \rho}{\partial x}} + \overline{v\dfrac{\partial \rho}{\partial y}} + \overline{w\left(\dfrac{\partial \rho}{\partial z} - \rho\gamma^*\right)}} \tag{13}$$

tends to zero, equation (10) reduces to

$$u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\left(\frac{\partial \rho}{\partial z} - \rho\gamma^*\right) = 0, \tag{14}$$

which expresses an exact balance between horizontal density advection and vertical density advection minus the term $w\rho\gamma^*$.

In what follows next we shall examine the general properties of flows that satisfy equations (7) and (14), which, as we have shown, hold in the limit when (*a*) Ω is so large that $\mathfrak{A} \to 0$, $\mathfrak{R} \to 0$ and $\mathfrak{E} \to 0$ (geostrophic limit), (*b*) Q is so small that $\mathfrak{Q} \to 0$ (isentropic limit) and (*c*) the relative fluid motions and their time variations are so slow that $\mathfrak{E} \to 0$ and $\mathfrak{D} \to 0$ (see equations (4), (6), (12) and (13)).

3. Steering

Now introduce the quantities $\mathbf{U}$ and ψ where

$$\mathbf{U} \equiv (u, v), \quad \psi \equiv \tan^{-1}(v/u), \tag{15}$$

respectively the horizontal flow velocity vector and the angle between $\mathbf{U}$ and the horizontal but otherwise arbitrary x axis, and $\mathbf{\Gamma}$, θ and γ where

$$\mathbf{\Gamma} \equiv \rho^{-1}\left(\frac{\partial \rho}{\partial x}, \frac{\partial \rho}{\partial y}\right), \quad \theta = \tan^{-1}\left(\frac{\partial \rho/\partial y}{\partial \rho/\partial x}\right); \quad \gamma \equiv \rho^{-1}\frac{\partial \rho}{\partial z}, \tag{16}$$

respectively the horizontal density gradient divided by ρ, the angle between $\mathbf{\Gamma}$ and the x axis, and the vertical density gradient divided by ρ. When expressed in terms of these quantities (7) and (14) are as follows:

$$\left\{\frac{\partial}{\partial s}\ln(\rho U), \frac{\partial \psi}{\partial s}, \frac{1}{\rho U}\frac{\partial(\rho w)}{\partial s}\right\} = \frac{g\Gamma}{2\Omega U}\{\sin(\psi-\theta), \cos(\psi-\theta), 0\} \tag{17}$$

and

$$U\Gamma\cos(\psi-\theta) + w(\gamma-\gamma^*) = 0. \tag{18}$$

Eliminating $\cos(\psi-\theta)$ between (18) and the second component of (17) we find for the rate of change with respect to s of the direction of the horizontal flow, $\partial\psi/\partial s$, the equation

$$\frac{\partial \psi}{\partial s} = -\frac{gw(\gamma-\gamma^*)}{2\Omega U^2}. \tag{19}$$

If we introduce as a convenient measure of the coherence of the flow patterns at different values of s the 'dimensionless steering parameter' or 'dimensionless axial coherence parameter'

$$\sigma \equiv 1/S\,\overline{\partial\psi/\partial s}, \tag{20a}$$

where S is a typical linear dimension of the fluid container in the direction of $\mathbf{\Omega}$, then by equation (19)

$$\sigma = \overline{2\Omega U^2/gwS(\gamma-\gamma^*)} \tag{20b}$$

for flows that satisfy (7) and (14). When σ is very large the horizontal flow *pattern* changes very slowly with s, notwithstanding the possibility of large gradients of the horizontal flow *speed* in the s direction.

When the fluid is homogeneous (i.e. $\nabla\rho = 0$), thermodynamic considerations are not required, so that $\mathfrak{Q}$, $\mathfrak{C}$, and $\mathfrak{D}$ are redundant parameters. Equation (17) then reduces to the celebrated Proudman–Taylor theorem $\partial\mathbf{u}/\partial s = 0$ (Proudman 1916, Taylor 1923, see also Greenspan 1968), with components

$$\partial U/\partial s = 0, \quad \partial\psi/\partial s = 0, \quad \partial w/\partial s = 0. \tag{21}$$

According to this theorem, geostrophic motion of a homogeneous fluid is such that fluid filaments parallel to $\mathbf{\Omega}$ move in directions perpendicular to $\mathbf{\Omega}$ without bending or stretching (and is therefore impossible unless the shape of the fluid container is fairly simple). Thus, the motion at all values of s is 'steered' by the topography of the container, an effect first demonstrated by Taylor (1923). By equation (5), when slight ageostrophic effects are taken into account the dimensionless steering parameter σ (see equation (20a)) is of order $(\mathfrak{A}+\mathfrak{R}+\mathfrak{E})^{-1}$ for a homogeneous fluid (cf. Lighthill 1970), and is therefore infinite in the geostrophic limit. The Proudman–Taylor theorem and 'Taylor columns'† produced by moving objects or topography are central concepts in the study of rotating homogeneous fluids.

Strong steering (i.e. $\sigma \gg 1$) also occurs even when $\Delta\rho$ is so large that

$$\partial \ln (\rho U)/\partial s \nless S^{-1},$$

provided that horizontal advection of density is so weak that

$$\overline{\Gamma U \rho \cos(\psi - \theta)} \ll \overline{2\Omega\rho U^2/gS} \tag{22a}$$

(see equation (17)). This criterion is satisfied in the case we are considering (namely $\mathfrak{Q} \to 0$, $\mathfrak{C} \to 0$, $\mathfrak{D} \to 0$) when the vertical advection of potential density satisfies

$$\overline{w\rho(\gamma - \gamma^*)} \ll \overline{2\Omega\rho U^2/gS}, \tag{22b}$$

but not otherwise.

4. Concluding remarks

Equation (1) is linear in $\mathbf{u}$ when $\mathbf{E}$ or $\mathbf{A}$ is the largest term on the right-hand side, but not when $\mathbf{R}$ dominates. Likewise equation (10) can be linearized when diffusion of density, as included implicitly in the term Q, is more important than non-linear density advection terms on the left-hand side, but not otherwise. It is hardly surprising therefore that rigorous mathematical studies of rotating fluids are almost exclusively concerned with hypothetical systems for which $\mathfrak{E}$ or $\mathfrak{A}$ (see equation (4)) is the leading measure of departures from geostrophy and $\mathfrak{Q}$ is so large that horizontal density advection can be treated as a small perturbation or neglected altogether. Within their limitations these detailed analyses provide useful insight into certain processes (Greenspan 1968, Veronis 1970) (and have only occasionally led to erroneous results as a consequence of misapplication).

In rapidly rotating large-scale natural systems and in many laboratory experiments, $\mathfrak{R}$ (rather than $\mathfrak{E}$ or $\mathfrak{A}$) is typically the leading measure of ageostrophic effects, irreversible thermodynamic processes act so slowly that $\mathfrak{Q} \ll 1$ (see equation (12)) and the parameters $\mathfrak{C}$ and $\mathfrak{D}$ (see equation (6)) are also much less than unity. By (5), $\overline{w}$ is then typically $\lesssim \overline{U}\mathfrak{R}S/L$ (where L is a typical horizontal dimension) and therefore much less than $\overline{U}$. Nevertheless, in systems that derive

† The use of terms with no precise complete definition out of context is not without precedent in fluid mechanics; 'eddy', 'vortex', 'blocking', 'wake' and 'wave' are a few examples. The expression 'Taylor column' was first coined (so far as I am aware) by Hide (1961) as a convenient term in the discussion of the flow phenomenon in Jupiter's atmosphere that, on the proposal that astronomers subsequently termed the 'Taylor column theory', underlies the Great Red Spot. The various phenomena to which the term 'Taylor column' has been applied by fluid dynamicists have in common two general characteristic features only: (*a*) they occur in fluids through which, owing to rapid rotation, strong stable density stratification, or magnetohydrodynamic effects, mechanical energy can be transmitted by transverse wave motions; (*b*) the corresponding value of σ (see equation (20*a*)) is much greater than unity. 'Taylor columns' exhibit wide variations in their other properties; they are not necessarily stagnant, they can occur in baroclinic as well as in barotropic fluids, and they can be produced by forced disturbances of the density or pressure fields as well as the velocity field.

their kinetic energy from potential energy due to gravity acting on the non-homogeneous density field (thermal convection) $|\overline{w}|$ is essentially non-zero. By (19) and (20) the dimensionless steering parameter satisfies

$$\sigma \gtrsim \mathfrak{B}^{-1}, \tag{23}$$

where

$$\mathfrak{B} \equiv g\overline{|\gamma-\gamma^*|}\,S^2/4\Omega^2L^2 \tag{24}$$

(cf. Prandtl 1952, Lineykin 1955, Stommel & Veronis 1957, Robinson 1960, Phillips 1963, Ingersoll 1969, Walin 1969).

Baroclinic waves

Experiments on thermal convection in a rotating fluid subject to a horizontal temperature gradient (see Hide 1969*b*) have shown that when Ω is sufficiently large for baroclinic waves (of the same type as the basic energy-producing motions in the Earth's atmosphere (see Phillips 1963)) to occur, the waves evidently 'choose' the largest possible scale in the s direction, namely S, and adjust their structure until the vertical stability, as measured by $\gamma-\gamma^*$, and the horizontal scale of the velocity and density fields, L, are such that $\mathfrak{B} \sim 1$. In such a system steering, as measured by σ (see equation (20)), is comparatively weak, for then by (23) $\sigma \sim 1$. This result is in keeping both with theoretical (Charney 1947, Eady 1949) and observational studies of growing baroclinic waves, which are characterized by vertical variations in horizontal phase of about 0·25 wavelength.

Jupiter's atmosphere

Motions in the atmosphere of the planet Jupiter are characterized by very low values of $\mathfrak{A}$, $\mathfrak{R}$ and $\mathfrak{E}$ and it is plausible to suppose that $\mathfrak{Q} \ll 1$, $\mathfrak{C} \ll 1$ and $\mathfrak{D} \ll 1$ (see Hide 1969*a*). For those levels within the atmosphere where magneto-hydrodynamic effects are unimportant, σ will be given by (23). The so-called 'Taylor column' theory of Jupiter's Great Red Spot (Hide 1961) requires that $\sigma \gg 1$ which by (24) implies that $\mathfrak{B} \ll 1$. If we take $g \sim 3\times10^3\,\mathrm{cm\,s^{-2}}$, $\gamma^* \sim 10^{-7}\,\mathrm{cm^{-1}}$ (see Tejfel' 1969), $S \sim 10^8\,\mathrm{cm}$ and $L \sim 10^{10}\,\mathrm{cm}$ then by (24) $\mathfrak{B} \sim \overline{(\gamma-\gamma^*)/\gamma^*}$. The sign of $\gamma-\gamma^*$, let alone its value, is not known for Jupiter's atmosphere, but values of $\overline{[(\gamma-\gamma^*)/\gamma^*]}^{-1}$ and therefore of σ that are not much greater than unity would be very hard to reconcile with certain observational evidence – notably that the planet emits a great deal more thermal radiation than it receives from the sun (see Trafton & Wildey 1970), implying that the atmosphere is probably subject to strong heating from below. Hide (1962) has already shown that the Taylor column theory of the Great Red Spot does not imply very special values for the vertical density gradient in Jupiter's atmosphere and the present analysis effectively generalizes this result to include effects due to horizontal density gradients.

Ocean currents

These are also characterized by low values of $\mathfrak{R}$, $\mathfrak{A}$, $\mathfrak{E}$, $\mathfrak{Q}$, $\mathfrak{C}$ and $\mathfrak{D}$. Vertical profiles of mean horizontal currents have been determined at a few stations in the Mediterranean (Swallow 1969, also private communication), from the motion of neutrally-buoyant floats at depths ranging from 100 m to 1500 m, and at one station in the Western Atlantic (Webster 1968), from moored current meters at depths ranging from 100 m to 2000 m. Because the horizontal scale of these currents greatly exceeds the depth of the ocean, vertical profiles should not differ significantly from profiles in the s direction. The most striking feature of these profiles is the smallness in both cases of vertical variations in the direction of $\mathbf{U}$. Typically these variations are no more than 11° for the Atlantic observations – notwithstanding a concomitant decrease of the magnitude of the current by a factor of 4 – the flow at all levels being nearly parallel to the large-scale features of the bottom topography. The corresponding ranges of directions and speeds revealed by the Mediterranean results are 15° and a factor of about 2 respectively. In both cases the dimensionless steering parameter σ is much greater than unity.

I am grateful to Dr M. E. McIntyre for helpful comments on an early draft of this paper and to the Director-General of the Meteorological Office for permission to publish it.

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